THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
Prof. E. T. WHITTAKER, M.A., F.R.S.

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The Mathematical Association.

THE ANNUAL MEETING will be held at the London Day Training College, Southampton Row, London, W.C. 1, on *Monday*, 1*st January*, 1923, at 5.30 *p.m.* (Advanced Section); and *Tuesday*, 2*nd January*, 1923, at 10 *a.m.* and 2.30 *p.m.* (Ordinary Meeting).

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

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VOL. XI. DECEMBER, 1922. No. 161.

APPROXIMATE INTEGRATION.*

BY ARNOLD BUXTON, M.A.

(*Continued from p.* 156.)

V. *Mechanical Integration.*

(*a*) *Planimeters.*—Hitherto we have considered the evaluation of areas and integrals by empirical approximations; we now have to consider the mechanical devices which exist for determining these quantities automatically. The most common instrument of this type is known as the planimeter, the theory of which is given now in most text-books, but usually in such a way that the important steps are not sufficiently emphasised. Only the geometrical

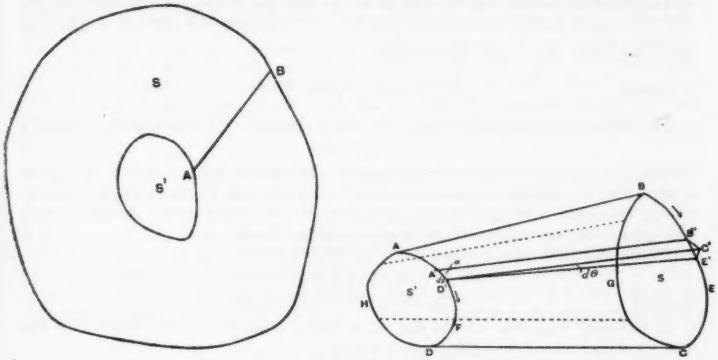

FIG. 3.

treatment will be considered as the analytical method is comparatively long and laborious. *The whole theory of the planimeter rests solely on the theorem that if a rod moves in a plane so that its ends describe closed curves S and S_1, then the area swept out by the rod is $S - S_1$.* The proof of this theorem is clear by inspection of the figure when the area S_1 is contained inside the area S. When the area S_1 is external to S, then from Fig. 3 the area *ABGCDFA* is

* A paper read to the London Mathematical Association, Nov. 16, 1921.

described both in the forward movement and in the backward movement of the rod, so that the area swept out by the rod is

$$BECG+BGCDFA-DHAF-BGCDFA$$

or

$$S-S'.$$

The next step is to obtain an expression for the area swept out by the rod between two consecutive positions. Its motion can be divided into two parts :

(1) a purely translational motion $A'B'$ to $D'C'$;
(2) „ rotational „ $D'C'$ to $D'E'$.

The area swept out in these two movements is

$lA'D' \sin\alpha$, where α is the angle $A'D'$ makes with $A'B'$
$+\frac{1}{2}l^2\,d\theta$, where $d\theta$ „ $C'D'B'$.

Hence

$$S-S'=\int l\,ds\sin\alpha+\tfrac{1}{2}\int l^2\,d\theta.$$

Writing $ds\sin\alpha=p$,

$$S-S'=l\Sigma p+\tfrac{1}{2}\int l^2\,d\theta.$$

Now the rod, in returning to its original position AB, either turns through no angle at all, in which $\int d\theta=0$, or makes n complete revolutions, in which case $\int d\theta=2n\pi$. Also the end A can be made to move to take the simplest cases either on a circle or straight line, so that S' is known, and therefore if we can measure Σp, we have a ready means of obtaining the area of any closed curve S. It is necessary then to introduce a device attached to the rod which will measure all and indicate the summation of the displacements of the rod perpendicular to itself. This can be achieved when there is attached to the rod at a distance c from one end, a graduated wheel, with its plane perpendicular to the length of the rod, and which can roll in contact with the plane of the diagram as the tracing end B of the rod traces out the required curve. For if the wheel records dw say in the movement of the rod from $A'B'$ to $D'E$, then $dw=p+c\,d\theta$ or $w=\Sigma p+c\int d\theta$.

Hence

$$S-S'=lw-lc\int d\theta+\tfrac{1}{2}l^2\,d\theta.$$

Therefore if the rod performs a cyclical movement with no resulting rotation,

$$S-S'=lw.$$

Now, if the end A moves along a curve or line and back again along the same curve or line to its starting point, while the end B traces out the closed curve, the area of which is required, then $S'=0$ and $S=lw$. Clearly then, the wheel can be graduated to read the area direct.

If the rod performs n complete revolutions, then

$$S-S'=lw-2lcn\pi+n\pi l^2,$$

so that

$$S=S'+lw-2lcn\pi+n\pi l^2.$$

If the end A moves on the arc of a circle, then $S'=n\pi a^2$, where a is the radius of the circle, so that

$$S=lw+n\pi(l^2-2lc+a^2).$$

(i) In the Amsler polar planimeter the *bar AB sweeping out the area* is

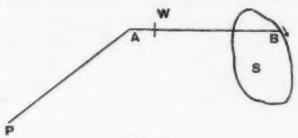

FIG. 4.

jointed to another rod AP and the *point P is fixed.* If there is no complete revolution of the bar as in the figure, then the locus of A as B describes S is clearly along the arc of a circle centre P and back again to its starting point. Hence $S'=0$ and $S=lw$. If the area of S is large and the pole P comes inside the area, then the rod AB describes a complete revolution, and

$$S=lw+\pi(a^2+l^2-2lc).$$

This latter term is the constant usually supplied with the instrument.

(ii) In the Hine-Robertson planimeter the end A moves along a straight line in a groove, so that as B goes round a closed curve and returns to its starting point, A moves along the groove and back again, so that S' is again zero. Amsler also adopts this movement of A along a straight line (hence the term line planimeter) in the Moment Integrator, but the Hine-Robertson, instead of using a graduated wheel to measure the movement of the rod perpendicular to itself, introduces a wheel with its perimeter in the form of a knife edge and with its plane always parallel to AB. This wheel can move freely along an arm perpendicular to AB, and as its lowest point is in contact with the paper, instead of slipping on the paper, it will slide along the bar.

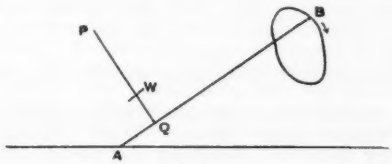

FIG. 5.

This sliding will clearly equal the roll of the wheel in the Amsler planimeter, and therefore the bar PQ can be graduated in terms of the area. As the instrument is made only small areas can be determined, and unfortunately the arm PQ is not at right angles to AB. I consider, however, that this type illustrates the theory of the planimeter much better than any other type, and I am having an instrument made for my own department which will be similar in principle but which will also be applicable for determining larger areas.

(iii) By far the most interesting instrument from the mathematical point of view is the Hatchet planimeter. The most complete theory of the instrument is to be found in Mr. F. W. Hill's paper to the *Phil. Mag.* 1894, and I understand that it was first used by boilermakers. It consists of a bar with a tracing point B at one end and a knife edge at the other, the plane of which contains the length of the bar as in the figure. As B traces out the curve the area of which is being sought, the knife edge A is always moving along the tangent to the curve traced out by A. The area swept out is as before,

$$S-S'=\int l\,ds\sin\alpha+\tfrac{1}{2}l^2\int d\theta,$$

with $\alpha=0$, since BA is always a tangent to S', so that $S-S'=\frac{1}{2}l^2\theta$, where θ is the inclination between AB at the start and at the finish.

Now, if S' can be made equal to zero as in cases (i) and (ii), then the value of S would be equal to $\frac{1}{2}$ length of AB multiplied by arc of circle radius l and centre the starting point B. It is clear that if the tracing end starts on the perimeter of the curve, then S' is never equal to zero except in the hypothetical case when B describes a circle radius BA. Hence it has been suggested that the start should be made at some point in the area, then down on the perimeter and round the perimeter back again to the starting point inside the

curve (see Fig. 6). That there is such a point in the general case which will make S' zero is undoubted, but so far as I am aware the problem of finding the exact location of this point has yet defied solution. From the diagram, however, it is shown that the area S' is much reduced by starting

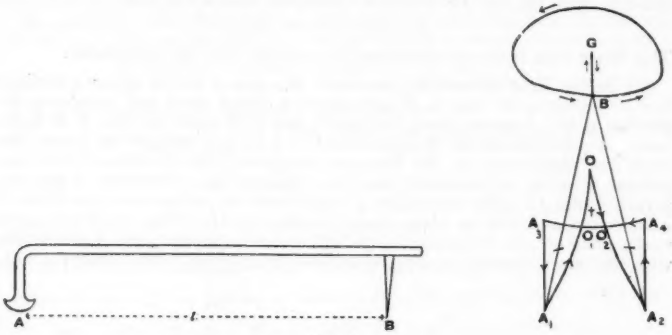

FIG. 6.
Showing the nature of S' curve (closed by arc of circle A_4A_3, centre G).

inside the area. The end A traces out area O_1OO_2 as positive area and $A_3A_1O_1$ and $A_4A_2O_2$ as negative areas. Hence $S' = O_1OO_2 - A_3A_1O_1 - A_4A_2O_2$. The instrument is very useful for illustrating the tractrix traced out in the problem of the master moving on a straight line, and his dog, on a lead of constant length, following him, and always moving towards him.

(iv) The remaining instruments chiefly represent more elaborate designs embodying the principles already discussed. There is, however, one exception in the case of the Rolling-sphere Planimeters. Taking the Line Rolling-sphere Planimeter first, it may have been noticed that area to be determined for a line planimeter is $\int l \sin\theta \, dx$, where θ is the inclination of the rod BA to the line traced out by A and taken as the axis. Now the mechanism

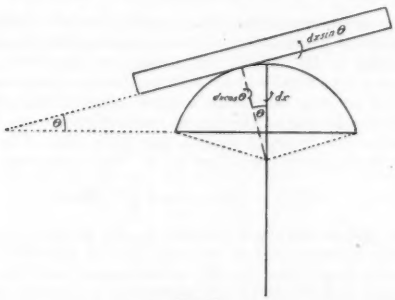

FIG. 7.

consists of a carriage guided by two heavy wheels in contact with the paper, and ensuring that while the end B traces out the closed curve, the instrument moves bodily only along the axis of x, *i.e.* A moves along a straight line and

the bar AB rotates about A. As the wheels move a distance dx, they actuate the rotation of a thin rod, one end of which is in the form of a portion of a sphere, which therefore rotates through an amount proportional to dx. This sphere is in contact with a small cylinder (see Fig. 7, the spherical portion being exaggerated to show the principle), the axis of which is always parallel to the bar AB. When this bar is at inclination θ, the axis of the cylinder is also at the same inclination; further, it is tangential to the sphere, so that when the latter rotates through an angle proportional to dx the cylinder will rotate about its own axis through an angle proportional to $dx \sin\theta$, since angular rotation is a vector. Hence the rotation of the cylinder, which is recorded on a graduated wheel, is a measure of the area. It can easily be shown that in the case of the polar planimeter, also, the area determined is $\int l \sin\theta \, dx$, so that the same sine-mechanism can be used. The only drawback is, that the comparatively small angle between the axis of the cylinder in its extreme positions necessarily limits the range of application of the instruments. Both instruments are well made and give good accuracy. The maker is Coradi of Zurich.

The Rolling-disc Planimeter is simply a more elaborate form of the Polar Planimeter, except that the wheel is no longer in contact with the paper but with a plane prepared surface, and thus the error introduced by creases in the paper are eliminated. Amsler also has various types of the Polar Planimeter, two of which are shown, one varies the length of the bar AB to suit various scales, square inches, square centimetres, etc., and the other carrying in addition a parallelogram link-work for use on small areas.

[*N.B.*—So far as I know there are no instruments in extensive use—although theoretically there is no reason why there should not be—which could be classified as rectangular coordinate and polar-coordinate planimeters—the names clearly denoting the problems they solve. The Hele-Shaw planimeters make use of the sine-mechanism described in (iv).]

(*b*) *Integrators.*—The term Integrators is usually applied to the type of instrument which determines the area, first moment and second moment, or moments of inertia of the area about a given axis. The Amsler integrator is about the only well-made instrument in use, and its theory is very simple.

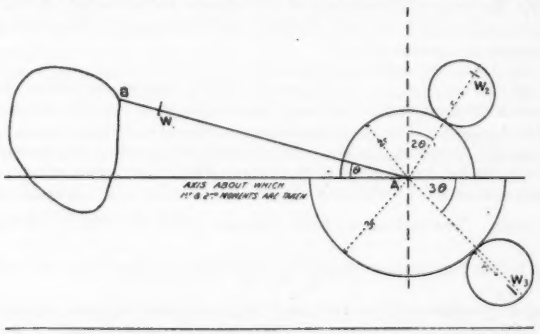

FIG. 8.

It is a combination of three planimeters, the first embodying the design of the line planimeter, and so determining the area of the curve (A_1, say). The other two determine similarly two other areas A_2 and A_3, which can be shown, however, to be connected with the first and second moments respectively about

the given axis (taken as the axis of x). The second recording wheel is in contact with the plane of the paper, but is embedded in a circular plate, which is geared, by a gear ratio of 2 to 1, to the frame of the instrument. Thus, when the arm AB moves through an angle θ, the line AW_2 moves through an angle 2θ, where 2θ is measured as shown in the figure. The third wheel is also embedded in a circular plate working on a gear ratio of 3 to 1, so that when AB turns through θ, AW_3 turns through 3θ, as shown in the figure.

Then, clearly, $$A_1 = \int l \sin\theta \, dx \quad (\text{or } \int y \, dx),$$

$$A_2 = \int l \cos 2\theta \, dx,$$

$$A_3 = \int l \sin 3\theta \, dx,$$

or $$A_2 = \int l \, dx - 2\int l \sin^2\theta \, dx$$

$$= 0 - 4/l \int y^2/2 \, dx$$

$$= 0 - 4/l \times \text{1st Moment}, \quad \textit{i.e. 1st Moment} = -lA_2/4.$$

Similarly

$$A_3 = \int 3l \sin\theta \, dx - 4\int l \sin^3\theta \, dx$$

$$= 3A_1 - 4 \times 3/l^3 \int y^3/3 \, dx$$

$$= 3A_1 - 12/l^3 \times \text{2nd Moment}, \quad \textit{i.e. 2nd Moment} = l^3A_1/4 - l^3A_3/12.$$

Now $$A_1 = lw_1, \quad A_2 = lw_2, \quad A_3 = lw_3,$$

i.e. $$\text{1st Moment} = -l^2w_2/4,$$

$$\text{2nd Moment} = l^3[3w_1 - w_3]/12,$$

or if $l = 8''$, as in the Amsler integrator, and each wheel makes 4 complete revolutions to 1", then

1st Moment $= -4 \times$ reading of 2nd wheel,

2nd Moment $= (32 \times$ reading of 1st wheel $- 32/3 \times$ reading of 3rd wheel).

[Note the calibration of the 3rd wheel has been slightly altered to make the lost constant 10 instead of $\frac{32}{3}$.]

And the Area $= 2 \times$ reading of 1st wheel.

If $l = 4''$, $A_1 = w_1$, $A_2 = -w_2$, $A_3 = 4w_1 - 4w_3/3$, and as any error clearly increases with the magnitude of l, the 4" tracer should be used whenever possible.

(*c*) *The Integraph.*—It remains to consider the type of instrument known as the Integraph, which traces out continuously the graph of the integral curve of a given curve, *i.e.* if $y = f(x)$ is the equation of a given curve, then the integraph will convert automatically the graph of $y = f(x)$ into the graph of $Y = \int f(x)dx$. The ordinates of the integral curve will clearly be the values of the $\int f(x)dx$ for the corresponding values of x, so that if we are considering the area $\int_0^X f(x)dx = \phi(X) - \phi(0)$, say, then the final ordinate of the integral curve will be $Y = \phi(X) - \phi(0)$.

The instrument is mounted on rollers so that it can only move bodily in one direction, say parallel to the axis of x. There are two fixed parallel bars AB and CD which are parallel to the y-axis (see Fig. 9) and the tracer T for the original curve slides along AB relative to the instrument for its y-motion and bodily with the instrument for its x-motion. As T moves along AB

it actuates a sliding bar TT_1 to rotate and slide about T_1, a fixed point, the position of which, however, can be varied by altering the constant (a) of the instrument. Let θ be the inclination of TT_1 to AB; then $\tan\theta = \frac{a}{y}$. The most important feature of the instrument is a wheel resting in contact with the

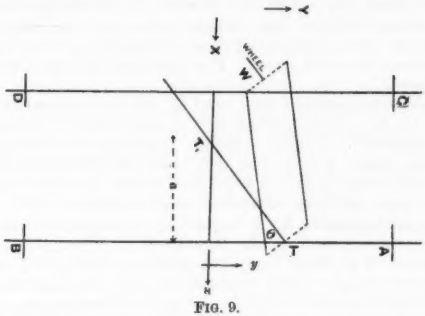

FIG. 9.

paper, with its plane vertical, and actuated by a parallelogram link-work motion, so that its direction is always parallel to TT_1. The locus of the point of contact of this wheel is the required integral curve. For if X, Y are the coordinates of a point on the integral curve,

$$\frac{dX}{dY} = \tan\theta = \frac{a}{y}.$$

Since the tracers move bodily with the instrument for their x-motion,

$$dX = dx, \quad i.e. \quad y\,dx = a\,dY,$$

and

$$\int y\,dx = aY,$$

so that the second tracer is clearly marking out the graph of $\int y\,dx$. The parallelogram link-work slides along CD as θ alters, thus ensuring that the plane of the wheel maintains the same inclination to AB as TT_1. For further details of the mechanism, see Prof. Henrici's article on "Instruments of Calculation" in the *Encyclopedia Britannica*. The instrument very conveniently traces out the shearing-force diagram from the load per unit run curve (provided this is continuous), and the bending-moment diagram from the shearing-force diagram. The maker of the instrument is Coradi, Zurich.

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The Technical College, Cardiff.

GLEANINGS FAR AND NEAR.

143. In July, 1838, William Edward Forster (42 years afterwards appointed Secretary for Ireland), then a wool-sorter in the mills of the Peases at Darlington, worked daily in his slip-paper cap from six in the morning to six at night with an hour for breakfast and 1½ hours for dinner. By the following September we find him asking his parents for Abbot's *Trigonometry*, Hamilton's *Conic Sections*, Lacroix's *Differential Calculus*, and, above all, Taylor's *Elements of Algebra*, so that he might employ his leisure (!) to some purpose. He was then twenty years old.

THE COMPLETE ANGLE AND GEOMETRICAL GENERALITY.*

By D. K. Picken, M.A.

1. The very great importance to Elementary Euclidean Geometry of the figure intermediate between the Straight Line and the Triangle appears to have been missed: viz. of the plane figure formed by two intersecting straight lines, each unbounded both ways. For this type of figure we use the name "Complete Angle."

If the two lines be denoted by l_1 and l_2, we may denote Complete Angles by (l_1, l_2) and (l_2, l_1); and we shall find it important to distinguish between these two "opposite" figures. Again, denoting the unbounded straight line through the points A and B by AB—and *not* distinguishing between AB and BA—we have such notation as (AB, CD) for Complete Angles. But we shall find that quite the best notation is the unqualified AOB—equivalent to (OA, OB)—for the Complete Angle formed by two straight lines, of which A, B are points respectively and O the intersection. Thus if $A_1, A_2, \ldots$ denote different positions of A on its line, etc., A_1OB_1, A_2OB_2, etc., denote one and the same Complete Angle. This we shall see to be an important fact.

The place of the Complete Angle in the geometrical system gives it priority of right to the unqualified three-symbol "expression." The ordinary Angle-figure, in which both lines are bounded one way by their intersection O, may be called for distinction a "Simple Angle" and denoted by $\angle^{1a} AOB$; the corresponding elementary (positive or negative) angle-quantity † by $\angle AOB$. The triangle may be denoted by $\triangle^{1a} AOB$; its area by $\triangle AOB$. Some such scheme of notation is necessary to sound scientific presentation of the facts of Elementary Geometry.

2. It will appear that nearly all the Angle propositions of Geometry can be most simply and most generally stated in terms of congruence of Complete Angles—expressed by

$$(l_1, l_2) \equiv (l'_1, l_2'), \quad AOB \equiv A'O'B', \ddagger \text{ etc.}$$

More particularly, the Angle propositions for which the familiar ambiguity "equal or supplementary" has been characteristic take a more accurate form, which resolves that ambiguity and exhibits the fact that it has operated as a barrier to the development of important geometrical theory. In this connection we note that the congruence $AOB \equiv A'O'B'$ implies that $\angle AOB$ and $\angle A'O'B'$ (positive or negative angles) are either (1) equal or (2) *differ* by straight angle (S);

$$i.e. \ \angle AOB \sim \angle A'O'B' = 0 \text{ or } S;$$

or, again, admitting all the possible trigonometrical angles,

$$\angle AOB = \angle A'O'B' + n \, . \, S; \ n \text{ integral.}$$

It is the fact that the second alternative has always been expressed as a sum (viz., ignoring sign of the angles, $\angle AOB + \angle A'O'B' = S$; *i.e.* "supplementary" angles), rather than as a difference, that has interfered with a proper development of the geometrical theory of Angle.

[*Note.*—In the development sketched out below, no diagrams are used. They are, in fact, quite unnecessary, if the fundamental ideas are clearly understood. Readers can supply what they consider helpful. *The methods are essentially quite general*; and, therefore, *the simplest figure which exhibits*

---

* This paper is an abridged form of a much longer paper on "The Euclidean Geometry of Angle," for which space could not be found in the *Gazette*.

† The important distinction between Angle-figure and angle-quantity can be usefully indicated by means of the capital and the small initial letters.

‡ This implies $BOA \equiv B'O'A'$; but $AOB \not\equiv B'O'A'$. See § 1.

all the essential facts may always be used. We shall see throughout that *the maximum of mathematical simplicity is consistent with maximum generality.*]

3. (i) The Angle-relations of the congruence of two triangles ABC, $A'B'C'$ are either

(1) $BAC \equiv B'A'C'$, $CBA \equiv C'B'A'$, $ACB \equiv A'C'B'$,

or (2) $BAC \equiv C'A'B'$, $CBA \equiv A'B'C'$, $ACB \equiv B'C'A'$.

These two types (in Plane Geometry) may conveniently be distinguished by the terms (1) "congruent" and (2) "contra-congruent." (So, again, "similar" and "contra-similar.")

(ii) For the Isosceles Triangle, such that $AB = AC$, the Angle-relation is

$$ABC \equiv BCA \text{ or } CBA \equiv ACB, \quad i.e. \ (BC, BA) \equiv (CA, CB).$$

(To see the precise bearing of those congruences, the sides of the triangle must be extended beyond the vertices.)

(iii) (1) The Bisectors of the Complete Angle AOB are the *two* lines OC such that $AOC \equiv COB$ (and $BOC \equiv COA$).

(2) And from the limiting case when A, O, B are collinear we have the fact that

$$OC \perp OA \text{ implies } AOC \equiv COA,$$

or $l_1 \perp l_2$,, $(l_1, l_2) \equiv (l_2, l_1)$.

(3) Conversely, if $AOB \equiv BOA$,

then either (1) O, A, B are collinear points

or (2) $OA \perp OB$.

(iv) The Right Angle relation of (iii) (2) may, further, clearly be extended thus:

If $l_1 \perp l_2$ and $l_3 \perp l_4$,

then $(l_1, l_2) \equiv (l_3, l_4)$.

4. The definition (§ 1) of the Complete Angle notation has the following important corollary:

If A, B, C, D *be* (coplanar) *points such that*

$$ABC \equiv ABD,$$

the three points B, C, D *are collinear.*

[*Note.*—The two Angle expressions differ only in the symbols C, D at one extreme; and the *points* C, D are collinear with that which belongs to the common middle symbol B.]

This is one of a group of very important Complete Angle propositions. See §§ 9, 14 below.

5. The geometrical facts underlying Trigonometrical addition of angles give the following important propositions:

(i) If $AOB \equiv A'O'B'$ and $BOC \equiv B'O'C'$,

then $AOC \equiv A'O'C'$.

And so, again, if AOB, BOC, ... $KOL \equiv A'O'B'$, $B'O'C'$, ... $K'O'L'$,

then $AOL \equiv A'O'L'$.

(ii) A precisely analogous proposition, with contra-congruence in place of congruence:

If $AOB \equiv B'O'A'$ and $BOC \equiv C'O'B'$, then $AOC \equiv C'O'A'$, etc.

(iii) By a combination of (i) and (ii), if $AOB \equiv COD$, and we use $BOC \equiv BOC$ as auxiliary, so that AOB, BOC are contra-congruent to DOC, COB, then

AOC is contra-congruent to DOB,

and therefore $AOC \equiv BOD$.

These theorems have a striking analogy to the fundamental Ratio-theorems for which the Euclid references are "*ex aequali*" and "*componendo.*" But see further § 7 below.

6. (i) The fundamental Angle-proposition of Parallels takes the following simple general form:

Parallel lines form, with any transversal, congruent corresponding Complete Angles; *i.e.* if $l_1 \| l_2$, and l any transversal, $(l, l_1) \equiv (l, l_2)$; and conversely.

And the following form is an important corollary:

(ii) *If A, B, C, D be* (coplanar) *points such that*

$$ABC \equiv DCB \quad (\text{and } BCD \equiv CBA),$$

then $AB \| CD$.

[The form $(AB, BC) \equiv (CD, BC)$ is the key.]

Note that also $ABD \equiv CDB$; $BAC \equiv DCA$; $BAD \equiv CDA$.

(iii) If coplanar lines l_1, l_2, l_3, l_4 are such that $l_1 \| l_3$ and $l_2 \| l_4$, then

$$(l_1, l_2) \equiv (l_1, l_4) \equiv (l_3, l_2) \equiv (l_3, l_4).$$

This is the Angle property of the Parallelogram—also expressible thus:

$$ABC \equiv DCB \equiv CDA \equiv BAD.$$

7. The propositions of § 5 may now be given important generalisation.

(i) If $(l_1, l_2) \equiv (l_1', l_2')$ and $(l_2, l_3) \equiv (l_2', l_3')$,

then $(l_1, l_3) \equiv (l_1', l_3')$.

And there is an obvious extension to two sets of n lines.

(ii) So for contra-congruence relations.

(iii) And from these,

$$\text{if } (l_1, l_2) \equiv (l_3, l_4), \quad \text{then } (l_1, l_3) \equiv (l_2, l_4).$$

These generalised analogues of the Ratio theorems are powerful geometrical instruments.

[*Note.*—We may regard the "alternated" form of the Parallels relation $(l, l_1) \equiv (l, l_2)$, viz. $(l, l) \equiv (l_1, l_2)$, as *defining* a use of the Complete Angle expression (l_1, l_2) in the limiting case of parallelism.]

(iv) By § 3 (iv), if $l_1 \perp l_2$ and $l_3 \perp l_4$,

$$(l_1, l_3) \equiv (l_2, l_4) \quad \text{and} \quad (l_1, l_4) \equiv (l_2, l_3).$$

8. (i) The Angle property of the Circle takes the following simple general form:

The necessary and sufficient condition that four (coplanar) *points A, B, C, D be concyclic is*

$$ABC \equiv ADC$$

(or $BCD \equiv BAD$ or $ABD \equiv ACD$, etc.).

It is easy to see that this is the proper expression of the "concyclic" relation. But a really satisfactory proof (which must be omitted from this summary) is not so obvious.

Cor. In particular, using § 3 (iv), if $AB \perp BC$ and $AD \perp DC$, *A, B, C, D* are concyclic.

(ii) If AD be the tangent at A to the circle ABC,

$$DAC (\equiv AAC) \equiv ABC, \text{ by § 4 and § 8 (i)};$$

and, conversely, $$ABC \equiv DAC$$

is the condition that AD be tangential to the circle ABC.

(iii) An obvious important corollary is that

$$ABC \equiv CDA$$

is the condition for congruent (but not coincident) circles ABC, ACD with common chord AC.

[*Note.*—In § 8 (i) $\angle ABC$ may be equal, but may not be (trigonometrically) "supplementary," to $\angle ADC$. In § 8 (iii) $\angle ABC$ may be supplementary, but may not be equal, to $\angle ADC$.]

9. On the basis of the results of §§ 3, 4, 6 and 8, *every* congruence $PQR \equiv UVW$, in which the Angle expressions are 3-permutations of the four symbols A, B, C, D, can be given a simple geometrical interpretation.

It is, of course, sufficient to consider the congruence of ABC with every other Complete Angle of the type in question. To take only one example, different from those of the sections in question:

If $ABC \equiv CAD$, then $CEA \equiv CAD$, if the circles ABC, ACE be congruent [§ 8 (iii)]; and, by § 8 (ii), AD touches circle ACE. Hence AD touches the circle through A, C congruent (but not coincident) with circle ABC.

10. Elementary Orthocentre theory can not (so far as the writer has been able to learn) be given *general* treatment on standard methods. The following proofs are typical applications of the new principles of generality.

(i) If AL, BM, CN be the "perpendiculars" of the triangle ABC, and B, C, M, N, therefore, concyclic, etc., let O be the intersection of lines BM, CN, and K that of lines AO, BC; then A, O, M, N are concyclic; $\therefore$ by §§ 4 and 8,

$$KAM \equiv OAM \equiv ONM \equiv CNM \equiv CBM \equiv KBM;^*$$

$$\therefore A, B, K, M \text{ are concyclic, and } AKB \equiv AMB;$$

$$\therefore AK \perp KB, \text{ and } K \text{ coincides with } L.$$

(ii) $BOC \equiv MON$—the same Complete Angle (§ 1)
$\equiv MAN \equiv CAB.$

Hence the circles ABC, OBC are congruent; and so, also, OCA, OAB congruent with them.

The simple familiar proposition that if the line AL intersect the circle ABC again at Q, L is the mid-pt. of OQ, is ordinarily given proof which is not general (*i.e.* not capable of statement without specific reference to a diagram). The Complete Angle proof is as follows: As in (i),

$$OBC \equiv OBL \equiv ONL \equiv CNL \equiv CAL \equiv CAQ \equiv CBQ;$$

$\therefore$ (§ 3) BC is a bisector of the Complete Angle OBQ and bisects OQ (which $\perp$ it).

11. Simson's Line theory provides the most characteristic application of these ideas (which were in fact arrived at in a discussion of the difficulty of obtaining a *general* proof of the elementary theorem of Simson's Line).

(i) If P be concyclic with A, B, C and PX, PY, PZ the perpendiculars to the lines BC, CA, AB, then, from the facts that B, C, X* are collinear, etc., and A, P, Y, Z concyclic, etc., we have (§§ 4 and 8),

$$PXY \equiv PCY \equiv PCA \equiv PBA \equiv PBZ \equiv PXZ;$$

$$\therefore X, Y, Z \text{ are collinear.}$$

(ii) If the line PX intersect the circle ABC again at R,

$$ARX \equiv ARP \equiv ABP \equiv ZBP \equiv ZXP \equiv ZXR;$$

$$\therefore (\S 6)\ AR \parallel ZX.$$

(iii) The proof that Simson's Line bisects OP can be similarly generalised.

(iv) The extension to "isogonals" may be made in the following terms:

If X', Y', Z', on the lines BC, CA, AB respectively, be such that

$$(BC, PX') \equiv (CA, PY') \equiv (AB, PZ'),$$

then X', Y', Z' are collinear points.

For, by hypothesis, $AY'P \equiv AZ'P$; $\therefore A, P, Y', Z'$ are concyclic, etc.; and the argument of (i) and (ii) is therefore again applicable.

[It will be observed that the term "isogonal" is given a proper precision, in terms of the Complete Angle.]

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* These chains of absolutely general congruences—exhibited in §§ 10, 11—are characteristic of the power of the method.

(v) The generalisation for a point P *not* on the circle may be made in the following stages:

(1) If the lines AP, BP, CP intersect the circle again at U, V, W, the triangles UVW, XYZ are *similar* (in the precise sense of § 3).

(2) The scale-ratio of this similarity $= D.\Pi : PA.PB.PC$, if D be the diameter of circle ABC, and Π the "power-area" of the point P with respect to that circle; and therefore $\delta.\Pi = PA.PB.PC$, if δ be the diameter of circle XYZ.

(3) $\triangle XYZ : \triangle ABC = \Pi : D^2$;

$\therefore \triangle XYZ \propto \Pi$ when P varies.

[*Note.*—It is interesting to consider these results for a variety of special positions of P.]

12. An important element of generality is introduced into Ratio theory. Only the briefest reference is here possible.

(i) (1) If a transversal intersect lines OX, OY at A, B and the bisector of the Simple Angle XOY at K,

$$AK : KB = OA : OB,$$

in which *signs are ascribed to* OA, OB by reference to the standard directions OX, OY of the given lines.

(2) $$\triangle OAB = \tfrac{1}{2}.\sin XOY.OA.OB,$$

in which *signs of angle, lengths and area* are taken into account.

(ii) (1) The general proof of the fundamental theorem of Projective Geometry at once follows—in the form

$$(AB : BC)/(AD : DC) = (\sin XOY/\sin YOZ)/(\sin XOW/\sin WOZ),$$

$ABCD$ being a variable transversal of the given pencil O, $XYZW$.

(2) The projective properties of the Circle are immediately deducible from § 8.

[*Note.*—In the generality of these Projective Geometry theorems—generality which is, of course, absolutely essential to their use—the main ideas of this paper have undoubtedly been implicit; but the elementary implications have, for some reason, not been tracked down. Where the false ambiguity "equal or supplementary" (quite false in this context, where the trigonometric principles are essential) has been employed, the consequent error (in sign of a sine) has been automatically corrected by double occurrence !]

(iii) One of the most interesting exercises in the ideas—probably not of intrinsic importance—is provided by the generalised Ptolemy's theorem, in which from four given points twelve different triangles are derived, with sides in a specific proportion. These are proved to consist of six pairs of *congruent* triangles; and the six non-congruent triangles are proved all *similar* (in the precise sense of § 3).

(iv) The right-angled triangle, divided by a perpendicular, is an elementary instance in which the two sub-triangles are *similar*, and are *contra-similar* to the original triangle.

13. The Complete Angle is obviously a valuable instrument for importing generality into the Angle properties of the Conic.

(i) The properties culminating in the theorem that

If the line P_1P_2 intersect directrix in D, and the tangents at P_1, P_2 intersect in T, SD and ST are the bisectors of Complete Angle P_1SP_2

can be given simple general proof.

(ii) Following a standard notation, the triangles SPG, PMS are *contra-similar*. And from this more precise form of a well-known property we get general proof of the theorem that tangent and normal at P are the bisectors of the Complete Angle SPS'.

(iii) (1) The generality of the proposition on the inclinations of two tangents to the focal radii of their intersection, T, has always been a crux. By the method of Complete Angle congruences we give general proof to it, in the precise form

$$(t_1, ST) \equiv (S'T, t_2) \quad \text{or} \quad P_1TS \equiv S'TP_2$$

—of which the "alternated" form exhibits the fact that either tangent may, of course, be associated with either focus.

(2) And the treatment of the corresponding facts for the Parabola is likewise general.

14. To sum up—the "Complete Angle" brings a new method to bear upon the Euclidean Geometry of Angle. The power of the method is epitomised in the congruence relations

$$(1)\ ABC \equiv ABD, \quad (2)\ ABC \equiv DCB, \quad (3)\ ABC \equiv ADC,$$

giving general expression to the geometrical relations

$$(1)\ B, C, D \text{ collinear}, \quad (2)\ AB \parallel CD, \quad (3)\ A, B, C, D \text{ concyclic};$$

and in the "*ex aequali*" and "*componendo*" relations of § 7 (the importance of which is concealed by the condensation of this paper).

The applications are practically coextensive with the relevance of the Angle concept in Geometry; and Complete Angle congruences almost entirely replace quantitative relations.

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144. "A nonius, or, as our author rather affectedly calls it, a vernier."—*Edin. Review*, 1806, ii. p. 97. (Review of Wollaston's *Phil. Trans.* paper, 1802.) [The "nonius," better known in later years as a "verñier," was invented by Pedro Nuñez (Nunes, Nunnius, Nonius), who was Professor of Mathematics at the University of Coimbra, and Cosmographo Major to King Emmanuel of Portugal. Nuñez described in 1542 an instrument of his invention for the division of arcs, but the instrument claimed as his own by Vernier in the volume given below and in his *La Construction, l'usage et les propriétés du quadrant de mathématique*, etc. (1634), is a very great improvement on the "nonius."]

The following is from the *N.E.D.*: Paul Vernier, French mathematician, 1580-1637, describes it in a tract on *Quadrant Nouveau de Mathématique*, 1631, erroneously called a Nonius. 1766. Instructions for Halley's Quadrant, 17. A scale of divisions graduated on the chamfered edge or sloped side of the index, which scale is called the Vernier. 1774. M. Mackenzie, Maritime Survey, 28. 1798. Phil. Trans. lxxxviii. 473. Pedro Nuñez, 1492-1577. De Crepusculis, 1542 A.D., contrivance for gradation of mathematical instruments. Vernier an improved form of this. 1750. Phil. Trans. xlvi.

145. Nicole, one of the authors of the famous *Logic of Port-Royal*, relates the following: "One day I told Madame de Longueville that I could prove that there are at least two people living in Paris with the same number of hairs on their heads. She asserted that I could never prove this without counting them first.

"My premisses are these. No head has more than 200,000 hairs, and the worst provided has one. Consider 200,000 heads, none having the same number of hairs. Then each must have a number of hairs equal to some number from 1 to 200,000 both included. Of course if any have the same number of hairs my bet is won. Now take one more person, who has not more than 200,000 hairs on his head. His number must be one of the numbers 1 to 200,000 included. As the inhabitants of Paris are nearer 800,000 than 200,000, there are many heads with an equal number of hairs."

SOME ISOCHORDIC CIRCLES RELATED TO THE TRIANGLE.

TAKING ABC as the triangle of reference, with the usual notation for its sides and angles, let $l=s-a$, $m=s-b$, $n=s-c$. Let I, I_1, I_2, I_3 be the centres of the inscribed and escribed circles DEF, $D_1E_1F_1$, $D_2E_2F_2$, $D_3E_3F_3$.

Through A, B, C draw parallels to the opposite sides forming the triangle $A_1B_1C_1$, so that AA_1, BB_1, CC_1 are the medians of the triangle ABC, and there are then ten triangles like ABB_1, ACB_1, etc., of the same area.

On B_1C_1 as diameter describe a circle cutting BA, CA produced in P and Q, and AB, AC produced if necessary in P_1, Q_1. Let R, S, R_1, S_1 be similarly found by drawing the circle on diameter C_1A_1, and T, U, T_1, U_1 by the circle on diameter A_1B_1.

Now $AF=s-a$ and $AP=a$.

$$\therefore\ FP=s.$$

Similarly FS, DR, DU, ET, EQ are each equal to s.

Therefore a circle with I as centre and radius $\sqrt{(r^2+s^2)}$ will pass through the six points P, Q, R, S, T, U.

Any circle which, like this one, cuts off equal chords on the sides of the triangle ABC, I have called an *isochordic circle*.

Again, $AE_3=s-b$ and $AT_1=b-c$.

$$\therefore\ E_3T_1=s-c=n.$$

$$E_3Q=AQ-AE_3=a-s+b=s-c=n.$$

Similarly S_1F_3, F_3P_1, D_3U_1, D_3R each $=n$.

$\therefore$ A circle with centre I_3 and radius $=\sqrt{(r_3^2+n^2)}$ will pass through Q, R, P_1, U_1, T_1, S_1.

Similarly circles with centres I_1 and I_2 will pass through S, T, Q_1, R_1, U_1, P_1 and U, P, S_1, T_1, R_1, Q_1; and their radii will be $\sqrt{(r_1^2+l^2)}$, $\sqrt{(r_1^2+m^2)}$.

In each case the triangle formed by two radii of a circle and one of its intercepts is equal to the area of ABC. This adds 12 more triangles each of the same area, making 22 in all.

Let S_1T_1, P_1U_1, Q_1R_1, the common chords of three of these circles, meet at their radical centre V, and let V_1, V_2, V_3 be the other radical centres.

Now $$AS=AB+BS=c+b=TC+CA=AT.$$

$\therefore$ ST is parallel to the external bisector of angle BAC.

And $$AS_1=BS_1-BA=b-c=CA-CT_1=AT_1.$$

$\therefore$ S_1T_1 is parallel to the internal bisector of angle BAC, and so is perpendicular to ST.

Similarly P_1U_1 is perpendicular to PU and Q_1R_1 to QR. Therefore V is the orthocentre of the triangle $V_1V_2V_3$.

Through A_1 draw YZ parallel to BC to meet AB in Y and AC in Z.

Then

$$(AS/SY)(YA_1/A_1Z)(ZT/TA)=(b+c)/(-b+c)(a/a)/(b-c)/(b+c))=-1.$$

Therefore the three points S, A_1, T are collinear, and A_1 lies on the line V_2V_3.

Similarly B_1, C_1 lie on V_3V_1 and V_1V_2.

Incidentally we have made 2 or 6 more triangles each equal to ABC.

Let S_1T_1 meet BC in J.

Then
$$BJ/JC=(BS_1/S_1A)(AT_1/T_1C)$$
$$=b/c$$
$$=BA_1/CA_1.$$

$\therefore$ A_1J is the bisector of the angle BA_1C, and is therefore parallel to AI.
$\therefore$ $A_1JVT_1S_1V_1$ is a straight line.

Similarly B_1V bisects angle ABC and C_1V bisects angle AC_1B. So V is the centre of the inscribed circle of triangle $A_1B_1C_1$, and its radius $=2r$.

Again
$$\begin{aligned} VS_1/S_1P_1 &= \sin S_1P_1U_1/\sin S_1VP_1 \\ &= \sin \tfrac{1}{2}B/\sin\{180^\circ - \tfrac{1}{2}(A+B)\} \\ &= \sin \tfrac{1}{2}B/\cos \tfrac{1}{2}C. \end{aligned}$$

Similarly
$$VT_1/Q_1T_1 = \sin \tfrac{1}{2}C/\cos \tfrac{1}{2}B.$$

Also
$$\begin{aligned} S_1P_1 &= S_1B - BA + AP_1 = 2n, \\ Q_1T_1 &= 2m. \end{aligned}$$

$$\begin{aligned} \therefore\ VS_1 \,.\, VT_1 &= 4\,mn \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C \\ &= 4lmn/s \\ &= 4\Delta^2/s^2 = 4r^2. \end{aligned}$$

Therefore the inscribed circle of triangle $A_1B_1C_1$ is the orthogonal circle of the three circles whose intercepts are $2l$, $2m$, $2n$.

Similarly
$$\begin{aligned} V_1S_1/PS_1 &= \cos \tfrac{1}{2}B/\sin \tfrac{1}{2}B, \\ V_1T_1/QT_1 &= \cos \tfrac{1}{2}C/\sin \tfrac{1}{2}C. \end{aligned}$$

$$\begin{aligned} \therefore\ V_1S_1 \,.\, V_1T_1 &= 4mn \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C \\ &= 4mns/l \\ &= 4\Delta^2/l^2 \\ &= 4r_1^2. \end{aligned}$$

And so the escribed circles of the triangle $A_1B_1C_1$ are the orthogonal circles of the other three sets of isochordic circles.

As most or all of these properties were first proved by trilinear coordinates, I will furnish the trilinear equations of the principal points, lines and circles, first of all premising that from the I_1 point of view a and α are both to be considered negative; so that:

$s = \tfrac{1}{2}(a+b+c)$	becomes	$\tfrac{1}{2}(-a+b+c)$ or l,
$l = \tfrac{1}{2}(-a+b+c)$	becomes	$\tfrac{1}{2}(+a+b+c)$ or s,
$m = \tfrac{1}{2}(a-b+c)$	becomes	$\tfrac{1}{2}(-a-b+c) = -n$,
$n = \tfrac{1}{2}(a+b-c)$	becomes	$\tfrac{1}{2}(-a+b-c) = -m$,

while Δ remains unaltered.

Thus, from that point of view, the chord intercepted on the sides of the triangle by any of these circles is equal to the perimeter of the triangle, and the radius of the inscribed circle is equal to $2\Delta/\text{perimeter}$.

The coordinates of P, U are given by the equations:
$$\begin{aligned} \alpha/\{b(c+a)\} = \beta/(-a^2) = \gamma/0 &= 2\Delta/abc, \\ \alpha/0 = \beta(-c^2) = \gamma/b(c+a) &= 2\Delta/(abc), \end{aligned}$$
and the equation of UP is
$$a^2\alpha + b(c+a)\beta + c^2\gamma = 0.$$

Similarly the coordinates of P_1, U_1 are given by the equations
$$\begin{aligned} \alpha/\{b(c-a)\} = \beta/a^2 = \gamma/0 &= 2\Delta/abc, \\ \alpha/0 = \beta/c^2 = \gamma/\{-b(c-a)\} &= 2\Delta/abc. \end{aligned}$$

The coordinates of V and V_1 are given by the equations
$$\begin{aligned} a\alpha/l = b\beta/m = c\gamma/n &= 2\Delta/s, \\ a\alpha/s = b\beta/(-n) = c\gamma/(-m) &= 2\Delta/l. \end{aligned}$$

The equations of the circles $PQRSTU$, $PS_1T_1R_1Q_1U$ are
$$abc\Sigma a\beta\gamma + \Sigma a\alpha\Sigma a^2(b+c)\alpha = 0$$
and
$$abc\Sigma a\beta\gamma = \Sigma a\alpha\{a^2(b+c)\alpha + b^2(a-c)\beta + c^2(a-b)\gamma\}.$$

The equations of the orthogonal circles are

$$abc\Sigma a\beta\gamma = \Sigma aa\Sigma a(b-c)^2a$$

and $$abc\Sigma a\beta\gamma = (\Sigma aa)\{a(b-c)^2\alpha + (c+a)^2b\beta + (a+b)^2c\gamma\}.$$

The circle with centre A and radius a has the equation:

$$abc\Sigma a\beta\gamma - (c^2b/3 + b^2c\gamma)\Sigma a\alpha + a^2(\Sigma aa)^2 = 0.$$

The circles $V_1V_2V_3$, V_2V_3V have equations

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma(4s-a)a^2\alpha = 0,$$

$$abc\Sigma a\beta\gamma + \Sigma aa(-(4l+a)a^2\alpha + (4l-b)b^2\beta + (4l-c)c^2\gamma) = 0.$$

The nine-point circle of the triangle $V_1V_2V_3$:

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma a^3\alpha = 0,$$

the inscribed circle of the triangle ABC

$$abc\Sigma a\beta\gamma - \Sigma aa\Sigma l^2a\alpha = 0,$$

the nine-point circle of the triangle ABC

$$2\Sigma a\beta\gamma - \Sigma aa\Sigma a\cos A = 0,$$

and the circumscribed circle, $\Sigma a\beta\gamma = 0$, complete a series of four generations of allied circles.

It is plain from the proposition that has been proved above that the orthogonal circles of the inscribed and escribed circles of the triangle ABC are the four perimetro-chordic circles of the triangle formed by joining the middle points of BC, CA and AB. Calling the points where these cut the sides of that triangle P_3, Q_3, R_3, S_3, T_3, U_3, we find that the coordinates are given by the equations

$$P_3, \quad a/(-b) = \beta/(a+c) = \gamma/b = 2\Delta/(2bc);$$

$$Q_3, \quad a/(-c) = \beta/c = \gamma/(a+b) = 2\Delta/(2bc).$$

Assuming the equation to be

$$\Sigma a^2bc\beta\gamma = \Sigma aa\Sigma Aa\alpha,$$

and determining it to pass through the points P_3, R_3, T_3, we obtain the equation

$$\begin{aligned} 4A\{a^2(b+c) + a(b^2+bc+c^2) + bc(b+c)\} \\ = -a^4(b+c) - a^3(b^2+bc+c^2) + a^2(b^3 - 2b^2c - 2bc^2 + c^3) \\ + a(b^4 - b^3c - bc^3 + c^4) + b^4c - b^3c^2 - b^2c^3 + bc^4, \end{aligned}$$

which, in spite of its unpromising appearance, is equivalent to

$$4A = -a^2 + (b-c)^2 = -4mn.$$

So the equation becomes

$$abc\Sigma a\beta\gamma + \Sigma aa\Sigma mnaa = 0.$$

And one of its fellows is

$$abc\Sigma a\beta\gamma + \Sigma aa(mnaa - msb\beta - nsc\gamma) = 0,$$

and the intercepted chords on the sides of that triangle are s and l.

All circles concentric with the inscribed and escribed circles of a triangle are isochordic. They can, however, be grouped in three classes.

(1) After the set we have already considered, we can take those whose intercepts are the same multiple of $2s$, $2l$, $2m$, $2n$; in other words those in which a chord as base with the corresponding centre as vertex give equal triangles.

(2) Those sets in which all the chords made by the circles of every set are equal.

(3) Those in which the chords subtend equal angles at the several centres, and in which the triangles formed by two radii and a chord are similar.

(1) If we consider the circle which makes an intercept XY ($=2\mu s$) on BC with centre I,

$$BY = \mu s + m, \qquad CY = \mu s - n,$$

and the coordinates of Y are given by the equations

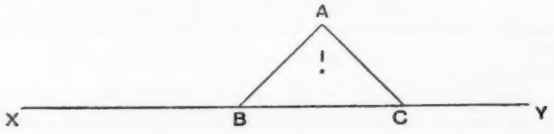
$$\beta/(-c(\mu s-n))=\gamma/b(\mu s+m)=\alpha/0=2\Delta/abc,$$

and the coordinates of X are given by the equations

$$\beta/c(\mu s+n)=\gamma(-b(\mu s-m))=\alpha/0=2\Delta/abc.$$

$\therefore$ The equation of the circle is :

$$\Sigma\{a^2\alpha^2(\mu^2s^2-l^2)+2bc\beta\gamma(\mu^2s^2+mn)\}=0.$$



Similarly changing the sign of a and α, the equation of the circle with centre I_1 and radius $2\mu l$ is seen to be

$$a^2\alpha^2(\mu^2l^2-s^2)+b^2\beta^2(\mu^2l^2-n^2)+c^2\gamma^2(\mu^2l^2-m^2)$$
$$+2bc\beta\gamma(\mu^2l^2+mn)-2ca\gamma\alpha(\mu^2l^2-ms)-2ab\alpha\beta(\mu^2l^2-ns)=0.$$

These equations can be put into the form

$$abc\Sigma a\beta\gamma+\Sigma a\alpha\Sigma(a\alpha(\mu^2s^2-l^2))=0$$

and $$abc\Sigma a\beta\gamma+(\Sigma a\alpha)(a\alpha(\mu^2l^2-s^2)+b\beta(\mu^2l^2-n^2)+c\gamma(\mu^2l^2-m^2))=0.$$

Similarly the circle whose centre is I_2 and radius $2\mu m$ is

$$abc\Sigma a\beta\gamma+(\Sigma a\alpha)(a\alpha(\mu^2m^2-n^2)+b\beta(\mu^2m^2-s^2)+c\gamma(\mu^2m^2-l^2))=0.$$

Hence the common chords of the circles of this series whose centres are I_1 and I_2 is

$$a\alpha(\mu^2(l^2-m^2)-s^2+n^2)+b\beta(\mu^2(l^2-m^2)+s^2-n^2)+c\gamma(\mu^2+1)(l^2-m^2)=0,$$

and the common chord of the circles whose centres are I_2 and I_3 is

$$a\alpha(\mu^2+1)(m^2-n^2)+b\beta(\mu^2(m^2-n^2)-s^2+l^2)+c\gamma(\mu^2(m^2-n^2)+s^2-l^2)=0.$$

So the coordinates of one radical centre are given by the equations

$$a\alpha/(\mu^2(2l-a)+2l+a)=b\beta/(\mu^2(2m-b)+2m+b)$$
$$=c\gamma(\mu^2(2n-c)+2n+c)=2\Delta/4s,$$

and the locus of these radical centres is

$$a\alpha(m-n)+b\beta(n-l)+c\gamma(l-m)=0$$

or

$$a\alpha(b-c)+b\beta(c-a)+c\gamma(a-b)=0.$$

Similarly the locus of the radical centre for the series, taking the three circles whose centres are I_2, I_3, I, is

$$a\alpha(b-c)+b\beta(c+a)-c\gamma(a+b)=0.$$

These are the straight lines joining I, I_1, I_2, I_3 to the centre of gravity of the triangle ABC.

If we make the centre of gravity the radical centre, we find $\mu^2=-\frac{1}{3}$.

If we make I the radical centre, we find $\mu^2=-1$.

The locus of the intersection of the circles of this series whose centres are I and I_1 is the circle

$$2(b+c)\Sigma a^2bc\beta\gamma-\Sigma a\alpha\{(b+c)(a^2+(b+c)^2)a\alpha$$
$$+(a^2+b^2-c^2)b^2\beta+(a^2-b^2+c^2)c^2\gamma\}=0,$$

and the centre of this circle is

$$a\alpha/\{-(b+c)(3a^2+(b+c)^2)\}=b\beta/\{b(a^2+3(b+c)^2)\}$$
$$=c\gamma\{c(a^2+3(b+c)^2)\}=2\Delta/\{2(b+c)((b+c)^2-a^2)\}.$$

This circle meets BC, where $b^2\beta=c^2\gamma$ and where $\beta\cos C=\gamma\cos B$.

And it meets AB and AC where the external bisector of angle A'

$$(b+c)a\alpha+b^2\beta+c^2\gamma=0$$

cuts them, and also where the parallel straight line

$$(a^2+(b+c)^2)a\alpha+(a^2+b^2-c^2)b\beta+(a^2-b^2+c^2)c\gamma=0$$

cuts them. These lines are both parallel to the external bisector of angle A, the latter through the point

$$a/0=b\beta/\cot B=c\gamma/(-\cot C).$$

The locus of the intersection of the circles whose centres are I_2 and I_3 is the circle

$$2(b-c)\Sigma a^2bc\beta\gamma$$
$$+\Sigma a\alpha\{(-b+c)(a^2+(b-c)^2)a\alpha-(a^2+b^2-c^2)b^2\beta+(a^2-b^2+c^2)c^2\gamma\}=0,$$

which is the same as the previous equation with signs of c and γ changed, so its centre is

$$a\alpha/\{(-b+c)(3a^2+(b-c)^2)\}=b\beta/\{b(a^2+3(b-c)^2)\}$$
$$=-c\gamma/\{c(a^2+3(b-c)^2)\}=2\Delta/\{2(b-c)/(b-c)^2-a^2)\},$$

and it meets BC, where $b^2\beta=-c^2\gamma$ and where $\beta\cos C=-\gamma\cos B$.

(2) An isochordic set of isochordic circles, where $2d$ is the length of the chord, will have the equations

$$abc\Sigma a\beta\gamma+\Sigma a\alpha\Sigma(a\alpha(d^2-l^2))=0,$$
$$abc\Sigma a\beta\gamma+\Sigma a\alpha(a\alpha(d^2-s^2)+b\beta(d^2-n^2)+c\gamma(d^2-m^2))=0.$$

The common chord of those with centres I_1 and I_2 will be the same for all values of d, *i.e.*

$$a\alpha(n^2-s^2)+b\beta(s^2-n^2)+c\gamma(l^2-m^2)=0$$

or
$$a\alpha(a+b)-b\beta(a+b)+c\gamma(a-b)=0,$$

which is the straight line parallel to the internal bisector of angle C through the middle point of AB.

And the radical centre of the circles with centres I_1, I_2, I_3 is given by the equations

$$a\alpha/(b+c)=b\beta(c+a)=c\gamma/(a+b)=2\Delta/4s=\tfrac{1}{2}r.$$

So every isochordic set of isochordic circles has the same set of radical centres as the inscribed and escribed circles.

The equation of the straight line joining the radical centres of the circles whose centres are I, I_2, I_3 and I, I_1, I_3, *i.e.* the points

$$a\alpha/(b+c)=b\beta/(c-a)=c\gamma/(b-a)=2\Delta/4l=\tfrac{1}{2}r_1$$

and
$$a\alpha/(-b+c)=b\beta/(c+a)=c\gamma/(a-b)=2\Delta/4m=\tfrac{1}{2}r_2,$$

is
$$a\alpha(a-b)-b\beta(a-b)+c\gamma(a+b)=0.$$

This is the straight line parallel to the external bisector of angle C through the middle point of AB, and the distance between the radical centres is divided at that point in the ratio $m:l$.

(3) When the triangles formed by two radii and a chord are similar:

In this case each intercepted chord is equal to 2μ times the radius of the corresponding inscribed or escribed circle, and the radius of the isochordic circle is $\sqrt{(\mu^2+1)}$ times such radius. Under these conditions the centres of similitude remain fixed for all the sets, and are the angular points of the triangle of reference, and the points where the internal and external bisectors of the angles A, B, C meet the opposite side and the circles of the different sets meet on the circle described on the line joining the two corresponding centres of similitude as diameters.

The equation of the circle whose centre is I is

$$\Sigma\{a^2\alpha^2(\mu^2r^2-l^2)+2bc\beta\gamma(\mu^2r^2+mn)\}=0.$$

And the equation of the circles which are the loci of the intersections of the circles whose centres are I and I_1, and I_2 and I_3, are

$$(b+c)\Sigma a\beta\gamma - \Sigma aa(c\beta\cos B + b\gamma\cos C) = 0$$

and

$$(b-c)\Sigma a\beta\gamma + \Sigma aa(c\beta\cos B - b\gamma\cos C) = 0$$

respectively, which are the circles on the internal and external bisectors of angle A as diameters.

The radical centre of the three circles whose centres are I_1, I_2, I_3 is given by the equations

$$\begin{aligned} aa/\{\mu^2 s(-lm^2 - ln^2 + m^2 n + mn^2) + lmn(2l + m + n)\} \\ = b\beta/\{\mu^2 s(-mn^2 - ml^2 + n^2 l + nl^2) + lmn(l + 2m + n)\} \\ = c\gamma/\{\mu^2 s(-nl^2 - nm^2 + l^2 m + lm^2) + lmn(l + m + 2n)\} \\ = 2\Delta/(4lmns) = 1/2\Delta, \end{aligned}$$

and the locus of these radical centres is

$$(b-c)a\cos A + (c-a)\beta\cos B + (a-b)\gamma\cos C = 0.$$

W. W. TAYLOR.

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146. (A youth) may have a strong genius for mathematics without being able to comprehend a demonstration of Euclid; because his mind conceives in a peculiar manner, and is so intent upon contemplating the object in one particular point of view, that it cannot perceive it in any other. We have known an instance of a boy, who, while his master complained that he had not capacity to comprehend the properties of a right-angled triangle, had actually, in private, by the power of his genius, formed a mathematical system of his own, discovered a series of curious theorems, and even applied his deductions to practical machines of surprising construction. – Goldsmith, p. 439, *Essays* xii.

147. I believe I was the first to recommend in parliament a uniformity of weights and measures, especially a decimal system, which eventually got the sovereign for the guinea. I was the mover of the new Board of Longitude, and of all chronometer-improvement measures. For many years I was on the Council of the Royal Society. I always tried to get the Northern powers to have a common thermometric scale, and got it talked of at the Congress of Vienna, and would probably have succeeded but for the escape of Buonaparte from Elba.—Croker to John Murray (*Letters*, 1851, iii. 243).

148. Learned he was in med'c'nal lore,
For by his side a pouch he wore
Replete with strange hermetic powder,
That wounds nine miles point-blank would solder;
By skilful chemist with great cost
Extracted from a rotten post. . . .

Arctophylax (Hudibras, Part. I. c. ii. 11, 223).

149. In mathematics he was greater
Than Tycho Brahe or Erra Pater;
For he, by geometric scale,
Could take the size of pots of ale;
Resolve by sines and tangents straight
If bread or butter wanted weight;
And wisely tell what hour o' th' day
The clock does strike, by Algebra. . . .

Hudibras, Part I. canto i. 11, 119.

SOME INCIDENTAL WRITINGS BY DE MORGAN.

(*Continued from* p. 163, vol. xi.)

XXXVI.

I. Ser. v. 89. JOHN OF HALIFAX.—Since every country has its *Holywood*, and *de Sacrobosco* does not distinguish Holy*wood* from Hali*fax*, John of Halifax has been claimed both by Ireland and Scotland, and, if I remember right, by some foreign countries. The manuscripts of his works, as well as the earlier printed editions, call him *Anglus* or *Anglicus*; and he lived in a time at which the natives of the three countries were as distinct as Frenchmen, Spaniards, and Italians. Bale, quoting Leland, calls him Halifax; as does Tanner: Pits gives his birth to Halifax. He was buried in the Maturin convent at Paris, where his epitaph existed in the sixteenth century. Pits implies that it appears from the epitaph that he died in 1256: Maestlinus expressly affirms that it can be collected from the epitaph, in the *Ad Lectorem* of his *Epitome Astronomiae*. All the authorities believe him to be English; and Leland thought he traced him as a student at Oxford. But had the manuscripts called him anything but English, the other evidence would not have weighed them down; for there are plenty of Holywoods, and there was, notoriously, a press of foreign students to Oxford in the thirteenth century. But name and residence in England may come in aid of the manuscripts.

The statement that he died in 1244 probably arises as follows. In the epitaph, according to Pits, are the following lines:

> "M. Christi bis C quarto deno quater anno
> De Sacrobosco discrevit tempora Ramus
> Gratia cui nomen dederat divina Johannis,"

meaning that in 1244 a bough from the holy wood *discrevit tempora*. This Pits calls an obscure reference to the time of his death, in the same sentence in which he places that time in 1256. Very obscure indeed, if a reference to his death in 1256 be intended. But if *discrevit tempora* refer, not to death, but to the matter of his celebrated work *de anni ratione, seu . . . computus Ecclesiasticus*, there is no obscurity at all. And at the end of a Merton manuscript of this *computus*, Tanner found the preceding lines inserted; the copyist taking them to allude, of course, to the date of the book. M.

XXXVII.

I. Ser. v. 103. QUERY ON THE CONTROVERSY ABOUT FLUXIONS.—In the Report made by the Committee of the Royal Society, it is stated that the Committee had "consulted the Letters and Letter-books in the custody of the Royal Society, and those found among the Papers of *Mr. John Collins* . . .;" thus leaving it doubtful whether Collins's papers *then* belonged to the Society, or, it may be, meaning to distinguish them as not so belonging.

In the preface to the *Analysis per Quantitatum Series* . . . by William Jones (father of his more celebrated namesake), London, 1711, 4to., which contains some of the matter published in 1712 in the *Commercium Epistolicum*, occurs the following passage:

"Etenim secundus jam agitur annus ex quo Scrinia *D. Collinsii* (qui, uti notum est, amplissimum cum sui saeculi Mathematicis commercium habuit) meas in manus inciderint; et in illis plurima reperi à cunctis fere totius *Europae* eruditis ipsi communicata; et inter ea non pauca, quae a Viro Cl. D. *Newtono* scripta fuerint."

This is hardly language which could be used with reference to papers lodged in the custody of the Society: it would seem as if Jones, in 1709 or 1710, became the owner or borrower of papers, till then in private hands exclusively. Can any evidence be brought forward as to the manner in which Jones and the Royal Society, or either, obtained these papers? I believe the Royal Society itself can give no information.

A. DE MORGAN.

XXXVIII.

I. Ser. v. 271. HINTS TO BOOK-BUYERS.—Inquirers buy books on subjects which they have, at the time, no particular intention of closely investigating: when such intention afterwards arises, they begin to collect more extensively. But it often happens, I suspect, that it does not come into their heads to examine what they have already got, as to which their memory is not good, because their acquisitions were not made under any strong purpose of using them. The warning which suggests itself is as follows: Always remember to examine the old library as if it were that of a stranger, when you begin any new subject, and before you buy any new books.

Here is another warning, not wholly unconnected with the former: Never judge of a book, that is, of all which comes between the two boards, by the title-page, which may be only the *first* title-page, in spite of the lettering at the back. Persons who bind their books will not always be bound themselves, either by law of congruity or convenience. I once hunted shop and stall for a speech delivered in parliament a century ago, not knowing that I had long possessed it bound up at the end of a Latin summary of Leibnitzian philosophy. At the risk of posthumously revealing my real name, I will add that I wrote on the fly-leaf that I was not the blockhead who bound the book. M.

XXXIX.

I. Ser. v. 276. JAMES WILSON, M.D.—In 1761 James Wilson, M.D., published in two volumes, octavo, a reprint of the mathematical tracts of his then deceased friend Benjamin Robins. To them he added an appendix containing a dissertation on the controversy about the invention of fluxions, which dissertation is very little cited. He makes various statements on his own authority, describing himself as having been the friend of Brook Taylor and of Dr. Pemberton. Among other things he furnishes something which might be cited in answer to my query in vol. v. p. 103, affirming that *all* Collins's papers fell into Jones's *possession* about the year 1708. Dr. Wilson and Martin Folkes were joint executors of Robins, as the former states. Query, Who was James Wilson, M.D.? What was his probable age in 1712? What means exist for forming an opinion as to his judgment and veracity, over and above his publications as aforesaid? A. DE MORGAN.

To this Mr. Bolton Corney contributed a reply (p. 329):

James Wilson, M.D.—To the numerous list of men whose services to literature our English biographers have injudiciously omitted to record may be added James Wilson, M.D. As editor of the *Mathematical Tracts* of Mr. Benjamin Robins in 1761, he has often been noticed with commendation. Beyond that circumstance, all is obscurity.

He wrote, however, a valuable *Dissertation on the rise and progress of the modern art of navigation*, which was first published by Mr. John Robertson in his *Elements of navigation* in 1764, and republished by him in 1772. The authors shall now speak for themselves:

"This edition (of the *Elements of navigation*) is also enriched with the history of the art of navigation; for, with the author's leave, I have published the following dissertation on that subject, written by Dr. Wilson, believing

it would afford the most ample satisfaction on that subject."—John Robertson, 1764.

"My enquiries into these matters (navigation) induced the late learned Dr. James Wilson to review and complete his observations on that subject, and produced his *Dissertation* on the history of the art of navigation, which he was pleased to give me leave to publish with the second edition of this work. . . . The second edition of these *Elements* having also been well received by the public, Dr. Wilson took the pains to revise his *Dissertation*, which he improved in many particulars."—John Robertson, Nov. 1, 1772.

"This *Dissertation*, written at first by desire, is now reprinted with alterations. Though I may be thought to have dwelt too long on some particulars, not directly relating to the subject; yet I hope that what is so delivered, will not be altogether unentertaining to the candid reader. As to any apology for having handled a matter quite foreign to my way of life, I shall only plead, that very young, living in a seaport town, I was eager to be acquainted with an art that could enable the mariner to arrive across the wide and pathless ocean at his desired harbour."—London, James Wilson, 1771 ?.

The united libraries of Henry Pemberton, M.D., F.R.S., and James Wilson, M.D., were sold in 1772. The sale occupied eighteen evenings, and produced £701 : 17 : 6. The learned writers, who were intimate friends, died within seven months of each other in 1771.

BOLTON CORNEY.

In a later communication (p. 362) Mr. Corney confesses that his statement as to the date of the sale of the combined libraries was mere guess-work, that further examination of manuscript and printed authorities led to some uncomfortable searchings of heart, but that ultimately he proved to be right "by chance." He proceeds as follows:

I had read the life of Pemberton in the *General biographical dictionary*. Chalmers therein states that his course of lectures on chemistry "was published in 1771 by his friend Dr. James Wilson." I applied for the volume at the British Museum. By a rare accident the *Scheme for a course of chemistry* was produced instead of the *Course of chemistry*, and as the day was far advanced, and *copy* due, I gave up the pursuit. On examination, it turns out that the volume contains a memoir of Pemberton in twenty-three pages. Chalmers cites Hutton and Shaw as his authorities; and *Hutton*, as I conceive, gives the substance of it as his own composition! Wilson, in this important memoir, declares that his intimacy with Pemberton was the *greatest felicity* of his life. He dates it the 10th Aug. 1771. He died on the 29th of September in the same year.

Wilson remarks in his previous work, that on the successful practice of navigation "depends, in an especial manner, the flourishing state of our country." To this remark no one can refuse assent. The *Dissertation* on the history of the art has fallen into oblivion, because it exists only in a work which has been superseded by others; but I venture to express my opinion that a separate edition of it, with such corrections and additions as might be required, and a continuation to the present time, would be a desirable addition to scientific literature; and that no one would perform the task with more ability, or more conscientiously, than professor De Morgan. BOLTON CORNEY.

[The reader will note Mr. Corney's omission of the usual capitals in titles of books. The indexes to "N. & Q." for these early years of its existence are not satisfactory examples of the art. It may be worth warning the reader that Wilson is incorrectly entered therein as John.]

? Was Robins published in later days ?

I. Ser. v. 399. JAMES WILSON, M.D.—This writer will be one instance of the use of such an organ of enquiry as "N. & Q." Mr. Corney's reply to my Query reminds me of Wilson's *History of Navigation*, with which

I have long been acquainted; but I had quite forgotten, or perhaps never remarked, that this Wilson was *James* and *M.D.* Baron Maseres reprinted the *History of Navigation* in the fourth volume of the *Scriptores Logarithmici*: it is an elaborate summary, of wide research, and puts the author's learning and judgment beyond a doubt. Maseres, in his Preface, gives a mention of Wilson, and, in addition to the facts now brought out, states, in his own curiously explicit style, that Dr. Pemberton's *Epistola ad Amicum J. W. de Cotesii inventis*, "was addressed to this Dr. James Wilson, who was the person meant by the word *Amicum*, with the two letters J. W., which were the initial letters of his name." I happen to possess Brook Taylor's copy of this *Epistola* (4to. 1722), and its Supplement (4to. 1723), in which Taylor has written, "E libris Br. Taylor, ex dono eximii paris amicorum, autoris D. H. Pemberton atque editoris D. J. Wilson." Thus it is established that the author of the dissertation on the fluxional controversy appended to Robins's tracts, lived in friendship with some of the most distinguished parties to that quarrel. It is also established that he was fully conversant with the mathematics of the day; for Pemberton's letter, called out by Wilson's own queries, could have been read by none but a previous reader of Cotes and the highest fluxionists. As to Wilson's age, he says (Robins's *Math. Tr.*, vol. ii. p. 299) he was a fellow-student of Pemberton at Paris: the latter was born in 1694, and the former was probably of nearly the same age. They were close friends to the end of their lives, and Wilson published Pemberton's *Course of Chemistry*, delivered at Gresham College, 8vo. 1771, according to Hutton and Watt. These last-named authorities both attribute to Pemberton himself the dissertation on the fluxional controversy in Robins's *Tracts*: but it certainly has Wilson's name to it; or rather, it is said to be by the *publisher* (which we now call *editor*) of the volumes. It is very likely that Pemberton gave help: assuredly he must have been consulted by his intimate friend on facts the truth of which was within his own knowledge. Accordingly, the following assertions, made by Wilson, are not to be lightly passed over: first (which also Robins assumes again and again), that *Newton* wrote the anonymous account of the *Commercium Epistolicum* (*Phil. Trans.* No. 342) usually attributed to Keill, which, in Latin, forms the Preface to the second edition of that work. Secondly, that Newton wrote the criticism on John Bernoulli's letter at the end of that second edition. Thirdly, that Newton himself, and not Pemberton, omitted the celebrated Scholium from the third edition of the *Principia*. Montucla, in the second edition (1802, vol. iii. p. 108) of his *History of Mathematics*, gives statements on these points from a private source, to the effect that the notes of the original edition of the *Comm. Epist.* were Newton's, and that the informant had seen the matter which was substituted for the Scholium, in Newton's handwriting, among the proof-sheets preserved by Pemberton. If Wilson were the informant, which may have been, for Montucla's first edition was published in 1758, Montucla must have confounded the two editions of the *Comm. Epist.* If not, it must have been some one who did not draw his account from the dissertation, in which there is nothing about the proof-sheets. Montucla, however, has lowered the credit of his informant, by making him assert that the second edition of the *Principia* was managed by Cotes and Bentley, without communication with Newton. This, which all the world knows to be untrue of the book, is true of the prefatory parts; and Wilson gives an account of Newton's dissatisfaction with those parts. If Wilson were the informant, Montucla has again misunderstood him. A. DE MORGAN.

(*To be continued.*)

MATHEMATICAL NOTES.

645. [A. 5. a.] *Partial Fractions associated with Quadratic Factors.*

1. In Professor Neville's note under the above heading in the *Mathematical Gazette* of January, 1922, on p. 10, it is stated that the most elementary proof of the possibility of resolving a rational fraction into partial fractions depends on the expression of the quotient of two factors which make up the denominator in the form of a simple continued fraction whose elements are polynomials. A full account of this method is given in the Fourth Edition of the First Part of Chrystal's *Algebra* in Chapters VI. and VIII.

I venture to suggest that the following method, which I use in teaching, and which, as far as I am aware, is not to be found in any of the text-books, is simpler, is more easily explained to students commencing the Integral Calculus, and that it will remove an intellectual difficulty which is slurred over in many treatises on Algebra.

Suppose that the fraction is denoted by $\frac{f(x)}{X^n\phi(x)}$, where $f(x)$, $\phi(x)$ are polynomials and X is a quadratic function of x; and that $f(x)$, $\phi(x)$ and X have no common factor, and that n is a positive integer.

Suppose further that the degree of the numerator is lower than that of the denominator. If this is not the case, then the denominator may be divided into the numerator, the polynomial quotient separated out, and the remainder of the division put in place of $f(x)$.

If the degree of $f(x)$ be l and that of $\phi(x)$ be m, then since X is a quadratic function, it follows that $l < 2n+m$.

The method described in Chrystal's *Algebra* depends on the expression of $X^n/\phi(x)$ in the form of a simple continued fraction.

2. Now
$$\frac{f(x)}{X^n\phi(x)} \equiv \frac{f(x)-(Px+Q)\phi(x)}{X^n\phi(x)} + \frac{Px+Q}{X^n}, \quad \text{.....(I)}$$

where P, Q are constants which will be determined in such a manner that the numerator of the first fraction on the right of (I) is divisible without remainder by X. This is the idea on which this demonstration is based.

Let $f(x)$, $\phi(x)$ be each divided by X.

Let the quotients be $g(x)$, $h(x)$ respectively and the remainders $Rx+S$, $Tx+U$ respectively.

So that
$$f(x) = Xg(x)+Rx+S,$$
$$\phi(x) = Xh(x)+Tx+U;$$
$$\therefore \frac{f(x)-(Px+Q)\phi(x)}{X^n\phi(x)} = \frac{g(x)-(Px+Q)h(x)}{X^{n-1}\phi(x)} + \frac{(Rx+S)-(Px+Q)(Tx+U)}{X^n\phi(x)}. \quad \text{.....(II)}$$

It remains only to show that *finite* values of P, Q exist such that
$$(Rx+S)-(Px+Q)(Tx+U)$$
is divisible without remainder by X, the quotient being independent of x.

Suppose $X \equiv ax^2+2bx+c$.

Then $(Rx+S)-(Px+Q)(Tx+U)$ is divisible by X if
$$-\frac{PT}{a} = \frac{R-(PU+QT)}{2b} = \frac{S-QU}{c}.$$

Solving these for P, Q, we get
$$\left.\begin{aligned} P(cT^2-2bTU+aU^2) &= a(RU-ST), \\ Q(cT^2-2bTU+aU^2) &= aUS-2bST+cRT. \end{aligned}\right\} \quad \text{.....(III)}$$

The values of P, Q will be *finite* if $cT^2-2bTU+aU^2$ do not vanish.
Now, if $cT^2-2bTU+aU^2=0$, then $-U/T$ is a root of $ax^2+2bx+c=0$.
$\therefore$ $Tx+U$ is a factor of $ax^2+2bx+c$.

But $$\phi(x)=(ax^2+2bx+c)h(x)+(Tx+U).$$

$\therefore$ $Tx+U$ is a factor of both $ax^2+2bx+c$ and $\phi(x)$.

This is contrary to the hypothesis that $ax^2+2bx+c$ $(=X)$ and $\phi(x)$ have no common factor.

In the only case of practical importance X is a quadratic expression with imaginary factors, say $X=a((x-k)^2+l^2)$, where a, k, l are real and a and l do not vanish.

In this case $$b=-ak, \quad c=a(k^2+l^2).$$

$$\therefore\ cT^2-2bTU+aU^2=a[(U+kT)^2+l^2T^2].$$

Now, if T do not vanish the expression $a[(U+kT)^2+l^2T^2]$ does not vanish, since a and l do not vanish.

On the other hand, if T vanish the expression reduces to aU^2. Since a does not vanish this could only vanish if U vanished. But then, if U vanish, since T also vanishes, $\phi(x)$ would be divisible by X without remainder, contrary to the hypothesis that X and $\phi(x)$ have no common factor.

Thus the values of P, Q given by (III) are finite.

Making use of these values,

$$\frac{Rx+S-(Px+Q)(Tx+U)}{X}=\frac{-PTx^2+(R-PU-QT)x+(S-QU)}{ax^2+2bx+c}=-PT/a.$$

$$\therefore\ \frac{Rx+S-(Px+Q)(Tx+U)}{X^n\phi(x)}=-\frac{PT}{a}\cdot\frac{1}{X^{n-1}\phi(x)}, \quad \text{.....(IV)}$$

and therefore (II) becomes

$$\frac{f(x)-(Px+Q)\phi(x)}{X^n\phi(x)}=\frac{g(x)-(Px+Q)h(x)}{X^{n-1}\phi(x)}-\frac{PT}{a}\cdot\frac{1}{X^{n-1}\phi(x)}.$$

$$\therefore\ \frac{f(x)}{X^n\phi(x)}=\frac{Px+Q}{X^n}+\frac{g(x)-(Px+Q)h(x)-\left(\frac{PT}{a}\right)}{X^{n-1}\phi(x)}. \quad \text{.....(V)}$$

The degree of the numerator of the second expression on the right of (V) is the greater of the numbers $l-2$, which is the degree of $g(x)$, and $m-1$, which is the degree of $(Px+Q)h(x)$.

The degree of $X^{n-1}\phi(x)$ is $m+2(n-1)$, which is greater than the degree of the numerator, since $l<m+2n$.

Hence the second fraction on the right of (V) is of the same form as the fraction on the left of (V), except that the exponent of X has been reduced by a unit. It can therefore be broken up in like manner, and thus the rational fraction is broken up step by step into fractions of the form $\frac{Px+Q}{X^{n-r}}$ and a single fraction whose denominator is $\phi(x)$.

3. The purpose of this note is simply to demonstrate the possibility of breaking up a fraction into partial fractions of the standard form. It does not pretend to deal with the practical determination of those partial fractions, which I take to be the main object of Professor Neville's note. It is not suggested that this method is preferable for practical use to any of those which Professor Neville describes.

University College, London. M. J. M. HILL.

Professor Neville writes:

"Most elementary" is ambiguous, and I am quite willing to withdraw the phrase. I was thinking of the logical sense in which that proof that $f/\phi\psi$ can be resolved is more primitive than any involving assumptions as to the form of one of the polynomials.

May I take the opportunity to point out that the method advocated in my note does establish incidentally the existence of the fractions? The essence of the method is the expression of $f(x)/X^n\phi(x)$ in the form

$$\{F'(X)=xF''(X)+X^nG(x,\ X)\}/X^n\{\Phi(X)+X^n\Psi(x,\ X)\}.$$

If $\quad F'(X)=L'+X^nU', \quad F''(X)=L''+X^nU'', \quad \Phi(X)=\Pi+X^n\Theta,$

where L', L'', Π are polynomials in X of degrees less than n, the division of L', L'' by Π as far as the term in X^{n-1} is *literally* the calculation of four polynomials Q', R', Q'', R'' such that identically

$$L'=\Pi Q'+X^nR', \quad L''=\Pi Q''+X^nR'',$$

and therefore such that

$$L'+X^nU'=\{\Pi+X^n(\Theta+\Psi)\}Q'+X^n\{R'+U'-(\Theta+\Psi)Q'\},$$
$$L''+X^nU''=\{\Pi+X^n(\Theta+\Psi)\}Q''+X^n\{R''+U''-(\Theta+\Psi)Q''\}.$$

That is, we have

$$\frac{f(x)}{X^n\phi(x)}=\frac{Q'(X)+xQ''(X)}{X^n}+\frac{\{R'+U'-(\Theta+\Psi)Q'\}+x\{R''+U''-(\Theta+\Psi)Q''\}+G}{\Phi+X^n\Psi},$$

and the mere form of the result shows that when X is expressed in terms of x the numerator of the last fraction is divisible by $(\Phi+X^n\Psi)/\phi$.

It is difficult to say what constitutes "practical importance" in such matters as these, but for simple integration it is not less desirable to keep X unresolved as long as possible if its factors are irrational than if they are complex.

To which Prof. Hill replies:

It seems to me that Professor Neville has read what I have written as involving a suggestion that the method he advocated did not prove the possibility of reducing the fraction in the usual way. I did not intend to suggest this.

My intention was to place at the disposal of the readers of the *Gazette* the simplest method of demonstrating the possibility of that reduction known to me, but apparently unknown to the writers of treatises on the subject.

My proof avoids:

(1) The use of the theory of continued fractions which, though leading to a valid proof, involves more knowledge than is usually possessed by those who are commencing the study of the Integral Calculus, and which is on that account avoided in elementary treatises;

(2) The assumption that, because the form of the partial fractions in common use contains as many disposable constants as there are linear simultaneous equations for their determination, the solution of those equations will lead to *finite* values of the constants. This assumption presents an intellectual difficulty to the careful reader, and it was the main object of my note to remove that difficulty in the simplest way.

646. [A. 1; 3.] The following demonstration of the Partial Fraction property affords a simple means of dealing with multiple roots.

To resolve $\dfrac{P(x)}{Q(x)}$ into Partial Fractions.

(1) Let $\qquad Q(x)\equiv(x-\alpha)^rR(x).$

Consider the expression

$$P(x)-\frac{P(\alpha)}{R(\alpha)}R(x)=(x-\alpha)\theta(x) \text{ by Remainder Theorem,}$$

where $\theta(x)$ is a polynomial which may or may not have $(x-\alpha)$ as a factor.

Dividing by $Q(x)$, $\dfrac{P(x)}{Q(x)}=\dfrac{P(\alpha)}{R(\alpha)}\dfrac{1}{(x-\alpha)^r}+\dfrac{\theta(x)}{(x-\alpha)^{r-1}R(x)}$,

and we repeat the process on 2nd term of right-hand side.

(2) This method may be extended to quadratic factors.

Suppose $Q(x)\equiv(ax^2+2bx+c)^rR(x)$,

and let α, β be the imaginary roots of $ax^2+2bx+c=0$; we shall first show that the values of A, B given by the equations

$$A\alpha+B=\frac{P(\alpha)}{R(\alpha)},\quad A\beta+B=\frac{P(\beta)}{R(\beta)},\text{ are real.}$$

Subtracting $$A(\alpha-\beta)=\frac{P(\alpha)}{R(\alpha)}-\frac{P(\beta)}{R(\beta)},$$

$\alpha-\beta$ is pure imaginary, as is $\dfrac{P(\alpha)}{R(\alpha)}-\dfrac{P(\beta)}{R(\beta)}$;

$\therefore$ A is real.

Again we easily obtain

$$B(\beta-\alpha)=\alpha\beta\left\{\frac{P(\alpha)}{\alpha R(\alpha)}-\frac{P(\beta)}{\beta R(\beta)}\right\};$$

$$\beta-\alpha,\quad \frac{P(\alpha)}{\alpha R(\alpha)}-\frac{P(\beta)}{\beta R(\beta)}\text{ are pure imaginaries,}$$

and $\alpha\beta$ is real; $\therefore$ B is real.

Suppose now A, B determined from these equations, and consider

$$P(x)-(Ax+B)R(x)=(ax^2+2bx+c)\theta(x)$$

by the Remainder Theorem.

$$\therefore\ \frac{P(x)}{Q(x)}=\frac{Ax+B}{(ax^2+2bx+c)^r}+\frac{\theta(x)}{(ax^2+2bx+c)^{r-1}R(x)},$$

and we may repeat the process.

13 Western Hill, Durham. J. L. BURCHNALL.

647. [I. 1.] *On Note* 611, xi. p. 24.

I think there is an error here, no doubt clerical.

In the *note* it should be $\log 9{\cdot}321=0{\cdot}9694.$

$$\text{antilog. } 0{\cdot}9694=9{\cdot}319.$$

Of course there is still a discrepancy with 4-figure tables.

With 5-figure tables we have

$$\log 9{\cdot}321=0{\cdot}96947.$$

antilog ·96947 = 9·3211, found as follows:

antilog ·9695 = 9·3217.

antilog ·9694 = 9·3196.

Diff. for ·0001 = ·0021.

„ ·00007 = ·0015 nearly.

$\therefore$ antilog. ·96947 = 9·3211.

At Cockburn High School, Leeds, this method has been in vogue for the last ten years. In Form III. the boys are taught in the very first stages of logarithmic computation to put their work down as follows:

$$\begin{aligned}x&=217{\cdot}6\times 9{\cdot}321\\&=10^{2\cdot3377}\times 10^{0\cdot9694}\\&=10^{3\cdot3071}\\&=2028.\end{aligned}$$

Of course similar working is gone through when division is performed, and also with powers and roots and more complicated calculations.

Pudsey, Nr. Leeds. ARTHUR B. OLDFIELD.

[Mr. F. J. W. Whipple remarks the notation is not new. "We put it in the *Twentieth Century Arithmetic*, and it was no novelty then."]

648. [V. a. c.] *The use of Geometry in elementary Algebra.*

When Algebra is used in elementary Geometry, it is primarily in order to make the proofs easier. The modern version is much easier than Euclid, Bk. II., which it replaces. But Geometry is used in elementary Algebra primarily as an aid to the understanding of the subject.

Consider two of the worst blunders which ignorance can perpetrate: the confusion of a^2 with $2a$ and the (alas, widespread) notion that the square of $a+b$ is a^2+b^2 or that the square root of a^2-b^2 is $a-b$.

To battle with the first by numerical substitution is no doubt desirable, but it is far more effective to draw a figure in which $2a$ is a length, and emphasise that a^2 is an area. It is easy to concoct questions about the perimeters and areas of squares and rectangles, and in particular the elusive facts that $(a/2)^2=a^2/4$ and $(a/3)^2=a^2/9$ may be brought home to the pupil by dividing a square into four or nine equal parts.

Cubes and cuboids will make clear the distinction between powers up to the third; beyond this, although we cannot picture, little difficulty will be felt if the first three have been mastered. Such a question as "the sum of the areas of the faces of a cube is $54a^2$ (or $3a^2/2$); find the volume and also the sum of the edges" is not easy at the appropriate stage, but will be found well worth the necessary effort.

Passing to the stage of the second blunder mentioned, the appropriate figures will naturally be drawn when the formula for $(a+b)^2$, $(a-b)^2$, and $(a+b)(a-b)$ are introduced, or by judicious arrangement the boys may find that they are doing this in Geometry about the same time, but it is also desirable that at some stage a good deal of algebraic drill should be done in connection with Pythagoras' theorem.

It should be possible for the boy to associate $\sqrt{(a^2+b^2)}$ *immediately* with the hypotenuse of a right-angled triangle of sides a, b. Once this association is established he is most unlikely to confuse it with the sum of the other sides. Similarly, if asked to prove that $a-b$ is less than $\sqrt{(a^2-b^2)}$ he should be able to do it either algebraically, since $(a+b)(a-b)>(a-b)(a-b)$, or from a figure, since the difference of two sides of a triangle is less than the third.

Further algebraic examples on Pythagoras' theorem will include finding the distances from the corner of a rectangle (sides a, b) to the mid points of the opposite sides, and similar questions concerning cuboids and perhaps cones and spheres.

Again, probably the best type of quadratic problem is to be found among the well-known questions in which the corners of a cardboard rectangle are cut off and the rest folded up to make an open box, or perhaps a box with a lid; at any rate these form a pleasant variant from those in which time is saved by an increase in speed or money by a drop in price.

Even a short note on Algebra would be incomplete without an allusion to graphs. The distinction between $2x$ and x^2 may perhaps be brought home to the pupil by drawing the graphs of these functions, but this is not the kind of help from Geometry which I wish to urge. In graphs a line is drawn to represent x^2, and the use of Geometry recommended above is that in which x is thought of as a length, x^2 as an area and x^3 as a volume. C. O. TUCKEY.

REVIEWS.

Elementary Vector Analysis with Application to Geometry and Physics. By C. E. WEATHERBURN. Pp. xxviii + 184. 12s. 1921. (Bell's Mathematical Series, Advanced Section.)

[We were looking forward to a review of this book by the late Prof. C. G. Knott, but no notes for it have been found among his papers.]

Dr. Weatherburn is known to readers of the *Gazette* as a firm believer in the use of vectors, and a text-book from him is very welcome. Since a knowledge of vectors may benefit many students who will never require the nabla or the dyadic, he has designed a volume complete in itself to cover the elementary parts of the subject, and here we have an admirable account of vector algebra and of differentiation and integration with respect to a single scalar variable, followed by a synopsis and preceded by a historical sketch that is well balanced and well written.

Perhaps even so brief a history as this should find room for the name of Joly, but our only serious complaint is that mention of two editions of Tait's *Quaternions* and of two of Gibbs-Wilson's *Vector Analysis* gives to the bibliographical information an air of completeness, doubtless unintentional, that is sadly misleading, since, to take the most striking examples, there is no reference to Joly's monumental edition of Hamilton's *Elements* (two volumes, 1899, 1901), to the 1878 reprint of the *Ausdehnungslehre von* 1844, to the current reprint of Gibbs-Wilson, or even, except incidentally as the source of a quotation, to the third edition (1890) of Tait's work, which differed much more from the second than the second from the first.

The elementary algebra of vectors is developed rapidly and clearly, with the usual convention that the measure of a vector is essentially positive, and with the usual neglect of some logical niceties : Hamilton (*Lectures*, p. 50 ; *Elements*, § 2) and Wilson (*Vector Analysis*, p. 6) were careful to say that the use of 0 for the zero vector was a new use of the old symbol. The system explained is that of Heaviside and Gibbs : the two primitive products are defined independently, the product that is scalar is the negative of Hamilton's scalar product, and there is no combination into a quaternion. It is generally conceded, we think, that this is the best system for beginners, but the pious may surely challenge the assertion that " imaginaries play a large part in Hamilton's theory ". The account of derivatives and integrals is tuned to precisely the right pitch.

The applications are to the geometry of lines, planes, spheres, and twisted curves, and to the mechanics of particles and rigid bodies, but not to physics in any broad sense. To quote from the Preface, " vectors were not designed for use in elementary plane geometry," and a learner is most likely to be attracted if the vectorial method is applied at once where the clumsiness of Cartesian machinery is most felt. In mechanics, statics is made to depend on dynamics. Dr. Weatherburn explains that this order, " not recommended for young beginners, is desirable for a student who is able to read this volume " ; we cannot believe that any student for whom this book is too hard will find three-dimensional mechanics profitable or even intelligible on any basis whatever, but there is so much to be said in favour of allowing dynamics to precede statics even in the most elementary stages that the apology is quite superfluous.

Perhaps the unit vectors **i**, **j**, **k** are mentioned too often : the learner is always too ready to run back to Cartesian coordinates, and will not appreciate that the ease with which he can go to them at any time is the very reason why there is no need to hurry there in a panic. The word ' module ' is ugly and unnecessary, and the symbol chosen for the unit vector of which a given vector is a positive multiple is neither expressive nor typographically convenient ; to call the two *vectors* by which a line is located Plücker's *coordinates* is mystifying. This is not the place in which to discuss the vexed question of notation for products, but we must protest against the printing of **a**•**bc** *without any gap between* **b** *and* **c** to denote the product of **c** by the number **a**•**b**, a vicious habit from which Gibbs and Wilson are free ; no logical error is

involved, but the practice is surely opposed to the spirit of vector analysis, which aims at reflecting in the grouping of symbols the intimacy of the relations between the magnitudes symbolised; also, to some writers **a.bc** is not a multiple of **c** at all, but is the multiple of **a** which Gibbs denotes by **a** $\mathbf{b}\cdot\mathbf{c}$, and confusion will become worse confounded if we have to accept not merely different groups of symbols for the same function but also incompatible interpretations of groups of symbols which differ if at all only in the level of a dot.

The synopsis, considering its unusual ratio to the matter summarised, might have been put to better use. This book is too short and is arranged too well for an elaborate synopsis to be of great value within the same covers, but if Dr. Weatherburn maintains the standard he reaches here, a summary of the complete work *issued as a supplementary volume* would have filled a place that is still vacant.

Lest our criticism of details cause a misunderstanding, let us conclude by saying emphatically that this is a good text-book, which should be useful to a much wider range of students than the author seems to have had in mind.

E. H. N.

An Introduction to Mathematical Analysis. By F. L. Griffin. Pp. viii + 512 + xxiii. 10s. 6d. net. No date. (Harrap.)

Calculus and Graphs. By L. M. Passano. Pp. viii + 167. 9s. net. 1921. (Macmillan.)

Differential Equations. By H. B. Phillips. Pp. vi + 78. 6s. 6d. net. 1922. (Chapman and Hall.)

These three American text-books seem to indicate that mathematical teachers in the United States work under conditions quite different from those that exist in England. We lay stress upon thoroughness and we exercise our pupils in grappling with rather difficult problems on a limited range of subjects. This training is very good for the ablest students, but for those who do not proceed further it involves a narrowness of outlook that is deplorable. The mathematical course that is confined to the requirements of such an examination as London Matriculation is quite unfitted to give a general idea of the power and importance of our science. In America, or at least in some parts of it, a wider range is covered and a broad general outlook obtained. The price that has to be paid for this is the sacrifice of much of the mental training obtained by working problems. Many of the exercises set for American students appear to us to be unduly easy.

Professor Griffin has drawn up a unified course that covers an immense amount of ground. A little geometry is supposed to be known, but the requirements in algebra are so modest that the book includes an account of the method of solving simple equations. Starting with this slender equipment, the student is introduced to graphs, differentiation, integration, trigonometry, logarithms, coordinate geometry, solution of equations, progressions, the binomial theorem, permutations, probability, the normal probability curve, the method of least squares, and complex numbers, in the order given here. The book would have been much improved if these subjects had been arranged in a more reasonable way. Still, even as it stands, the book does seem likely to achieve its object in giving a bird's-eye view of mathematics, showing that it can be applied to many branches of human activity. There are plenty of examples, mostly of a very simple character. If time permitted, it would do our pupils good to work rapidly through this book, provided that they afterwards took a more thorough course of the usual English kind.

Professor Passano's book is a very short introductory course on calculus, with a little about graphs. The applications of calculus receive special attention. The account of indefinite integration is very slight. Perhaps it is hardly fair to criticise the bookwork when the author states that it is his policy to sacrifice logical detail to simplicity of presentation. Limits are used without being defined, and no proof is given of the rule for differentiating powers of x other than positive integral ones. But further brevity and simplicity could have been obtained by omitting the questionable statement

(p. 13) that " Zero does not mean nothing " and the long and unconvincing treatment of the definite integral (pp. 88-92), in which small quantities are " neglected " in such a wholesale fashion that the author feels constrained to add a footnote to repress the scepticism which he expects this procedure to arouse. All this trouble could have been avoided by giving the simple treatment contained in Art. 145 of the 3rd edition of Hardy's *Pure Mathematics*. However, in spite of these defects, the book will be useful to those who desire to learn rapidly a little about the processes and applications of differentiation and integration.

The chief fault of Professor Phillips' *Differential Equations* is its extreme brevity. It contains less on differential equations than is given at the end of the usual English works on Integral Calculus. On the other hand it has a splendid collection of problems dealing with mechanics, physics, and chemistry. In these the student has first to form a differential equation from the data given and then integrate it. Several articles in the text illustrate the methods to be employed, but some of the problems, especially the electrical ones, are not likely to be solved by anyone restricted to the information given here. To those who have sufficient preliminary knowledge of the sciences involved the book should appeal strongly.

The method used for finding a particular integral of a linear equation with constant coefficients may be unfamiliar to English readers. Four rules are given (two of them not quite completely), the general principle of which is to assume a trial solution involving linearly all the functions that occur in the given differential equation and their derivatives. Exceptional cases arise when some of these functions occur in the complementary function. This method, which is used by several American writers, considerably reduces the amount of bookwork.

H. T. H. Piaggio.

MATHEMATICAL ASSOCIATION.

YORKSHIRE BRANCH.

A meeting of the Yorkshire Branch of the Mathematical Association was held at Leeds University on Saturday, Nov. 11. Professor Milne, on resigning the Presidency in accordance with the rules of the Branch, proposed as his successor Mr. W. F. Beard of Wakefield Grammar School, who was elected unanimously. Dr. Milne, whose work in founding the Branch has been so successful, was nominated as Vice-President with Dr. Clement Jones of Bradford. In place of the members of Committee who retire after three years' service, there were elected Mr. W. Newbold, Inspector of Secondary Schools, Mr. S. Lister of West Leeds High School, and Mr. W. H. Berwick of Leeds University. Mr. S. Lister, who during the war had experience as an Instructor of Midshipmen in Mathematics, gave a paper on " Some Problems of Navigation," after which Miss Sykes of Chapel Allerton High School introduced a discussion on the syllabus of the Additional Mathematics paper at the School Certificate Examination. As a result of this a Committee was appointed to co-operate with the Manchester Branch, which has also been discussing the matter, in making suggestions for a more suitable syllabus. A novel feature of the meeting was an exhibition of two rare first editions, one a copy of the first printed Euclid, the other a copy of the first edition of Copernicus' epoch-making work.

Obituary.

C. G. Knott, D.Sc., F.R.S.

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THE Library has now been removed to 29 Gordon Square, London, W.C. 1, and Mr. W. E. Paterson has taken over the duties of Honorary Librarian.

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Also Mr. J. V. H. Coates, who has presented:

Nova Acta Regiae Societatis Scientiarum Upsaliensis. IV. Series. Vol. V. Fasc. 1.

Laplace. *Mécanique Céleste*, translated, with commentary by Nathaniel Bowditch, F.R.S. 1729-39. 4 vols. roy. 4to. Boston.

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Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and has exerted an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

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